Tautological bundles on Hilbert schemes

X = sm. proj. variety / $\mathbb{C}$ (mostly dim = 1 or 2)

$X^{[n]}$ = Hilbert scheme of 0-dim length n subsch. of X

Fact: If $n \leq 3$ or $\dim X \leq 2$, $X^{[n]}$ is smooth.

$\Phi_{X,n}$ = universal subscheme of $X^{[n]} := \{x, Z \in X \times X^{[n]} : x \in Z\}$
(or Φ)

$X^{(n)}$ = nth symm. product of X (unordered n-tuples of points)

Hilbert-Chow morphism:

$$\rho : X^{[n]} \longrightarrow X^{(n)}$$

Set-theoretically, ρ "forgets" scheme structure.
In fact, it is a morphism of schemes (see FGA explained for construction of ρ.)

For each n, we have maps

$$\begin{tikzcd}
\text{univ. subsch. (incidence variety)} \arrow[r] & \Phi_n \subseteq X \times X^{[n]} \arrow[dl, "q_n"'] \arrow[dr, "\sigma_n"] & \\
X & & X^{[n]}
\end{tikzcd}$$

Let L be a line bundle on X.

Define $E_{n,L}$ (or E_L) $:= \sigma_* q^* L$

Note: Since σ is flat, E_L is a v.b. of rank n.

Define $N_{n,L} := \det E_{n,L}$

Basic Properties

- Fiber of $E_{n,L}$ over $Z = H^0(\sigma_* q^* L \otimes \mathcal{O}_Z) = H^0(L \otimes \mathcal{O}_Z)$

- $H^0(E_{n,L}) = H^0(q^* L) = H^0(L \otimes q_* \mathcal{O}_\Sigma)$

  projection formula

  ($= H^0(L)$ when q has conn. fibers, e.g. $\dim X = 1, 2$)

  Stein factorization

- Always have ev. map $H^0(L) \longrightarrow E_{n,L}$ (fibers: $H^0(L) \to H^0(L|_Z)$)
  surjective $\Longleftrightarrow$ L imposes indep. conditions on n-tuples of pts
  $\Longleftrightarrow$ L is "$(n-1)$-v. ample" (0-v.a. = b.p.f, 1-v.a. = v.a.)

- $H^0(N_{n,L}) = \Lambda^n H^0(L)$ (more later)

Why are they interesting?

- Natural way to get v.b. / l.b. on $X^{[n]}$
- Applications to secant varieties
- N_L gives embedding into Grassmannian for $L \gg 0$.
- Appl. to syzygies of L (Ein–Lazarsfeld)

Example

Let $X = \mathbb{P}^1$

$(\mathbb{P}^1)^{[n]} = (\mathbb{P}^1)^{(n)} = \mathbb{P}^n$

(n-tuples of pts on $\mathbb{P}^1 \longleftrightarrow$ n-forms in x, y up to scaling $\cong \mathbb{P}^n$)

Let $L = \mathcal{O}_{\mathbb{P}^1}(b)$ $\quad b \geq n-1 \Rightarrow \mathcal{O}(b)$ $(n-1)$-very ample

$$\begin{tikzcd}
\text{(P, n-form vanishing at P)} \arrow[r] & \Phi \subseteq \mathbb{P}^1 \times \mathbb{P}^n \arrow[dl, "q"'] \arrow[dr, "\sigma"] & \\
\mathbb{P}^1 & & \mathbb{P}^n
\end{tikzcd}$$

$\Phi \subseteq \mathbb{P}^1 \times \mathbb{P}^n$ is a divisor.

In $\mathbb{P}^1 \times \mathbb{P}^n$:

If $F \in \mathbb{P}^n$, then $\Phi \cap (\mathbb{P}^1 \times \{F\})$ = length n subscheme where F vanishes

If $P \in \mathbb{P}^1$, $\Phi \cap (\{P\} \times \mathbb{P}^n)$ = homogeneous forms vanishing along P
= hyperplane

So we have SES on $\mathbb{P}^1 \times \mathbb{P}^n$:

$$0 \longrightarrow \mathcal{O}(-n,-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\Phi} \longrightarrow 0$$

Twist by pullback of L:

$$0 \to \mathcal{O}(b-n,-1) \to \mathcal{O}(b,0) \to q^*\mathcal{O}_{\mathbb{P}^1}(b) \to 0$$

pushforward to $\mathbb{P}^n$:

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^1}\underset{\substack{\uparrow\\ \text{base change,}\\ \text{projection formula}}}{(b-n))} \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}\underset{\substack{\uparrow\\ \text{base change}}}{(b))} \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow E_L \underset{\substack{\nearrow\\ (n-1)\\ \text{-v.a.}\\ \Rightarrow \text{surj.}\\ \text{(no higher direct image)}}}{\longrightarrow} 0$$

follows that $N_L = \mathcal{O}_{\mathbb{P}^n}(b-n+1)$

Exercise: Fill in details.

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N_L in general

Lemma (Voisin – "Standard fact")

$$H^0(\underset{\substack{\uparrow\\ \text{locus of subschemes on sm. curve}}}{X^{[n]}_{curv}}, N_{n,L}) = \Lambda^n H^0(X, L)$$

(True on $X^{[n]}$ if $X^{[n]}$ is norma.)

Proof for curves: X a sm. proj. curve, L a l.b. $X^{[n]} = X^{(n)}$

ev: $H^0(L) \to E_{n,L}$ induces map $\Lambda^n H^0(L) \to H^0(N_{n,L})$.

Need to construct inverse:

$\varphi : X^n \longrightarrow X^{(n)}$ quotient by S_n-action.

$p_i : X^n \longrightarrow X$ i^{th} projection

There is a natural map $\varphi^* E_L \longrightarrow \oplus_i \, p_i^* L$

Exercise: Construct this map using formal properties.

On fibers over $(x_1, \ldots, x_n)$:
$H^0(L \otimes \mathcal{O}_Z) \rightarrow \oplus_i \, H^0(L \otimes \mathcal{O}_{x_i})$ restriction

Isomorphism off the diagonal, drops rank along divisor D supp. on Δ.

$\Rightarrow$ taking determinants, we get

$$0 \rightarrow \varphi^* N_L \rightarrow L^{\boxtimes n} \rightarrow L^{\boxtimes n} \otimes \mathcal{O}_D \rightarrow 0$$

$$\Rightarrow \varphi^* N_L \cong L^{\boxtimes n}(-D)$$

$$\begin{tikzcd} H^0(N_L) \arrow[rr] \arrow[dr] & & H^0(\varphi^* N_L) \\ & H^0(\varphi^* N_L)^{S_n} \arrow[ur] & \end{tikzcd}$$

is injective (φ is surjective)

In fact, factors like this

$$\left(\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr] & X \arrow[d] \\ & X^{(n)} \end{tikzcd} \quad \text{take pullbacks} \right)$$

$$H^0(\varphi^* N_L)^{S_n} \cong H^0(\varphi_* \varphi^* N_L)^{S_n} \cong H^0(\varphi_* \mathcal{O}_{X^n} \otimes N_L)^{S_n} \underset{\substack{\uparrow \\ \text{check on open} \\ \text{affines}}}{\cong} H^0((\varphi_* \mathcal{O}_{X^n})^{S_n} \otimes \underset{\substack{\uparrow \\ \text{already } S^n \\ \text{invariant}}}{N_L})$$

$\cong H^0(\mathcal{O}_{X^{(n)}} \otimes N_L)$

acts by permuting and sign (induced by determinant) ↙

So $H^0(N_L) = H^0(\varphi^* N_L)^{S_n} \hookrightarrow H^0(L^{\boxtimes n})^{S_n} \cong [\otimes_i H^0(L)]^{S_n}$

↑ check fiber-wise

can check: $\cong \Lambda^n H^0(L)$

↑ exercise

Left to check: $\Lambda^n H^0(L) \to H^0(N_L) \hookrightarrow \Lambda^n H^0(L)$

is the identity

<u>Idea of proof for higher dim X:</u>

$X^{(n)}_* :=$ n-tuples supported on $\geq n-1$ points

$X^{[n]}_*$, X^n_* inverse images of $X^{(n)}_*$

$\dim(X^{[n]}_{curv} \setminus X^{[n]}_*) \leq 2$, so global sections extend.

$$\begin{tikzcd}
B^n_* \arrow[r] \arrow[d] & X^n_* \arrow[d] \\
X^{[n]}_* \arrow[r] & X^{(n)}_*
\end{tikzcd}$$

$B^n_* = X^n_* \times_{X^{(n)}_*} X^{[n]}_*$

<u>Claim</u> $B^n_* \to X^n_*$ is blowup along $\Delta = \{(x_i) : x_i = x_j \text{ some } i \neq j\}$ and $B^n_* \to X^{[n]}_*$ is quotient by S_n-action.

<u>Idea</u>: Similar argument works as in case of curves by working on B^n_* instead of X^n.

Exercise: Generalize proof.

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N_L and Grassmannians

Let L be an $(n-1)$-v. ample line bundle on X (sm. variety)

Denote $\mathbb{G} = Gr(n, H^0(L)) = n$-dim quotients of $H^0(L)$

ev: $H^0(L) \longrightarrow E_{n,L}$ is surjective and induces map

$$\begin{aligned} \varphi : X^{[n]} &\to \mathbb{G} \\ Z &\longmapsto \left(H^0(L) \twoheadrightarrow H^0(L \otimes \mathcal{O}_Z)\right) \end{aligned}$$

So if $H^0(L) \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow Q \longrightarrow 0$ is univ. quotient,

univ. property of Grassmannians says ev is pullback of univ quotient.

So

$$\varphi^*\big(\underset{\substack{\uparrow \\ \text{Plücker embedding}}}{\Lambda^n} Q\big) = N_{n,L}$$

So we get a commutative diagram

$$\begin{tikzcd}
X^{[n]} \arrow[r, "\varphi"] \arrow[dr, "N_{n,L}"'] & \mathbb{G} \arrow[d, "\det Q"] \\
 & \mathbb{P}(\Lambda^n H^0(L))
\end{tikzcd}$$

$\Lambda^n H^0(L) \twoheadrightarrow H^0(N_{n,L}) \Rightarrow$ map is determined by complete linear system. Since it's an isomorphism, it's non-degenerate.

<u>Note</u>: φ is injective $\Longleftrightarrow$ no two length n subschemes determine the same n-plane $\Leftrightarrow L$ n-very ample

In fact, in this case φ is an embedding

Roundabout argument:

$$\begin{tikzcd}
\mathbb{P}(E) \arrow[r] \arrow[dr, "\text{Terracini: Birational onto image (away from } X)"'] & \mathbb{P}(Q) \arrow[d] \\
 & \mathbb{P}(H^0(L))
\end{tikzcd}$$

So $\mathbb{P}(E) \to \mathbb{P}(Q)$ is an embedding $\Rightarrow \varphi$ is an embedding