

Tautological bundles on Hilbert schemes

$X = \text{sm. proj. variety} / \mathbb{C}$ (mostly $\dim = 1$ or 2)

$X^{[n]} = \text{Hilbert scheme of 0-dim length } n \text{ subsch. of } X$

Fact: If $n \leq 3$ or $\dim X \leq 2$, $X^{[n]}$ is smooth.

$\mathbb{I}_{X,n} = \text{universal subscheme of } X^{[n]} := \{x, Z \in X \times X^{[n]} : x \in Z\}$
(or \mathbb{I})

$X^{(n)} = n^{\text{th}} \text{ symm. product of } X$ (unordered n -tuples of points)

Hilbert-Chow morphism:

$$\rho: X^{[n]} \rightarrow X^{(n)}$$

Set-theoretically, ρ "forgets" scheme structure.

In fact, it is a morphism of schemes (see FGA explained for construction of ρ .)

For each n , we have maps

$$\begin{array}{ccc} \text{univ. subsch.} & \rightarrow & \mathbb{I}_n \subseteq X \times X^{[n]} \\ \text{(incidence variety)} & \searrow^{q_n} & \swarrow_{\sigma_n} \\ X & & X^{[n]} \end{array}$$

Let L be a line bundle on X .

Define $E_{n,L}$ (or E_L) := $\sigma_* q^* L$

Note: Since σ is flat, E_L is a v.b. of rank n .

define $N_{n,L} := \det E_{n,L}$

Basic Properties

- Fiber of $E_{n,L}$ over $z = H^0(\sigma_* q^* L \otimes \mathcal{O}_z) = H^0(L \otimes \mathcal{O}_z)$
- $H^0(E_{n,L}) = H^0(q^* L) = H^0(L \otimes q_* \mathcal{O}_\Sigma)$
↙ projection formula
(= $H^0(L)$ when q has conn. fibers, e.g. $\dim X = 1, 2$)
Stein factorization
- Always have ev. map $H^0(L) \rightarrow E_{n,L}$ (fibers: $H^0(L) \rightarrow H^0(L|_z)$)
 surjective $\iff L$ imposes indep. conditions on n -tuples of pts
 $\iff L$ is "($n-1$)-v. ample" (0-v.a. = b.p.f., 1-v.a. = v.a.)
- $H^0(N_{n,L}) = \bigwedge^n H^0(L)$ (more later)

Why are they interesting?

- Natural way to get v.b. / l.b. on $X^{[n]}$
- Applications to secant varieties
- N_L gives embedding into Grassmannian for $L \gg 0$.
- Appl. to syzygies of L (Ein-Lazarsfeld)

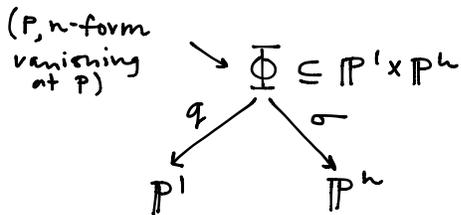
Example

Let $X = \mathbb{P}^1$

$$(\mathbb{P}^1)^{(n)} = (\mathbb{P}^1)^{(n)} = \mathbb{P}^n$$

(n -tuples of pts on $\mathbb{P}^1 \leftrightarrow n$ -forms in x, y up to scaling $\cong \mathbb{P}^n$)

Let $L = \mathcal{O}_{\mathbb{P}^1}(b)$ $b \geq n-1 \Rightarrow \mathcal{O}(b)$ $(n-1)$ -very ample



$\bar{\Phi} \in \mathbb{P}^1 \times \mathbb{P}^n$ is a divisor.

In $\mathbb{P}^1 \times \mathbb{P}^n$:

If $F \in \mathbb{P}^n$, then $\bar{\Phi} \cap (\mathbb{P}^1 \times \{F\}) =$ length n subscheme where F vanishes

If $P \in \mathbb{P}^1$, $\bar{\Phi} \cap (\{P\} \times \mathbb{P}^n) =$ homogeneous forms vanishing along P
 $=$ hyperplane

So we have SES on $\mathbb{P}^1 \times \mathbb{P}^n$:

$$0 \rightarrow \mathcal{O}(-n, -1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\bar{\Phi}} \rightarrow 0$$

Twist by pullback of L :

$$0 \rightarrow \mathcal{O}(b-n, -1) \rightarrow \mathcal{O}(b, 0) \rightarrow q_1^* \mathcal{O}_{\mathbb{P}^1}(b) \rightarrow 0$$

pushforward to \mathbb{P}^n :

$$0 \rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(b-n)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(b)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow E_L \rightarrow 0$$

\uparrow base change, projection formula \uparrow base change $\begin{matrix} (n-1) \\ \text{-v.a.} \\ \Rightarrow \text{surj.} \\ \text{(no higher direct image)} \end{matrix}$

follows that $N_L = \mathcal{O}_{\mathbb{P}^n}(b-n+1)$

Exercise: Fill in details.

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N_L in general

Lemma (Voisin - "Standard fact")

$$H^0(X_{\text{curv}}^{[n]}, N_{n,L}) = \wedge^n H^0(X, L) \quad (\text{True on } X^{[n]} \text{ if } X^{[n]} \text{ is normal.})$$

\uparrow locus of subschemes on sm. curve

Proof for curves: X a sm. proj. curve, L a l.b. $X^{[n]} = X^{(n)}$

$$\text{ev}: H^0(L) \rightarrow E_{n,L} \text{ induces map } \wedge^n H^0(L) \rightarrow H^0(N_{n,L}).$$

Need to construct inverse:

$$\varphi: X^n \rightarrow X^{(n)} \text{ quotient by } S_n \text{-action.}$$

$p_i : X^n \rightarrow X$ i^{th} projection

There is a natural map $\varphi^* E_L \rightarrow \bigoplus_i p_i^* L$

Exercise: Construct this map using formal properties.

On fibers over (x_1, \dots, x_n) :

$$H^0(L \otimes \mathcal{O}_Z) \rightarrow \bigoplus_i H^0(L \otimes \mathcal{O}_{x_i}) \text{ restriction}$$

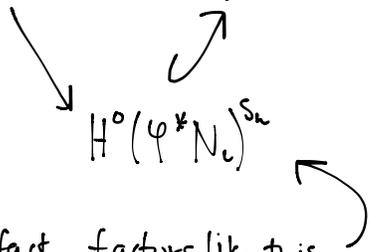
Isomorphism off the diagonal, drops rank along divisor D supp. on Δ .

\Rightarrow taking determinants, we get

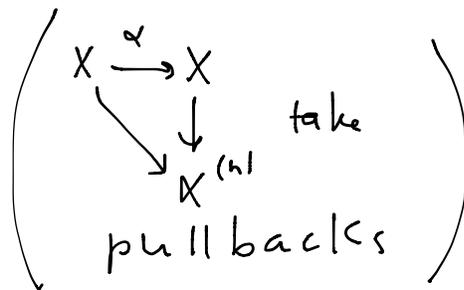
$$0 \rightarrow \varphi^* N_L \rightarrow L^{\boxtimes n} \rightarrow L^{\boxtimes n} \otimes \mathcal{O}_D \rightarrow 0$$

$$\Rightarrow \varphi^* N_L \cong L^{\boxtimes n}(-D)$$

$H^0(N_L) \rightarrow H^0(\varphi^* N_L)$ is injective (φ is surjective)



In fact, factors like this



$$H^0(\varphi^* N_L)^{S_n} \cong H^0(\varphi_* \varphi^* N_L)^{S_n} \cong H^0(\varphi_* \mathcal{O}_{X^n} \otimes N_L)^{S_n} \cong H^0((\varphi_* \mathcal{O}_{X^n}) \otimes N_L)^{S_n}$$

\uparrow check on open affine
 \uparrow already S^* invariant

$$\cong H^0(\mathcal{O}_{X^{(n)}} \otimes N_L)$$

So $H^0(N_L) = H^0(\varphi^* N_L)^{S_n} \hookrightarrow H^0(L^{\otimes n})^{S_n} \cong [\otimes_i H^0(L)]^{S_n}$ acts by permuting and sign (indexed by determinant)

\uparrow can check: $\cong \Lambda^n H^0(L)$

check fiber-wise \uparrow exercise

Left to check: $\Lambda^n H^0(L) \rightarrow H^0(N_L) \hookrightarrow \Lambda^n H^0(L)$
is the identity

Idea of proof for higher dim X:

$X_*^{(n)}$:= n-tuples supported on $\geq n-1$ points
 $X_*^{[n]}, X_*^n$ inverse images of $X_*^{(n)}$

$\dim(X_{\text{unr}}^{[n]} \setminus X_*^{[n]}) \leq 2$, so global sections extend.

$$\begin{array}{ccc} B_*^n & \longrightarrow & X_*^n \\ \downarrow & & \downarrow \\ X_*^{[n]} & \longrightarrow & X_*^{(n)} \end{array} \quad B_*^n = X_*^n \times_{X_*^{(n)}} X_*^{[n]}$$

Claim $B_*^n \rightarrow X_*^n$ is blowup along $\Delta = \{(x_i) : x_i = x_j \text{ some } i \neq j\}$
and $B_*^n \rightarrow X_*^{[n]}$ is quotient by S_n -action.

Idea: Similar argument works as in case of curves by working on B_*^n instead of X_*^n .

Exercise: Generalize proof.

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N_L and Grassmannians

Let L be an $(n-1)$ -v. ample line bundle on X (sm. variety)

Denote $G = Gr(n, H^0(L)) = n$ -dim quotients of $H^0(L)$

ev: $H^0(L) \rightarrow E_{n,L}$ is surjective and induces map

$$\begin{aligned} \varphi: X^{[n]} &\rightarrow G \\ \zeta &\longmapsto (H^0(L) \twoheadrightarrow H^0(L \otimes \mathcal{O}_\zeta)) \end{aligned}$$

So if $H^0(L) \otimes \mathcal{O}_G \rightarrow \mathcal{Q} \rightarrow \mathcal{O}$ is univ. quotient,

univ. property of Grassmannians says ev is pullback of univ quotient.

So

$$\varphi^*(\Lambda^n \mathcal{Q}) = N_{n,L}$$

\uparrow
Plücker embedding

So we get a commutative diagram

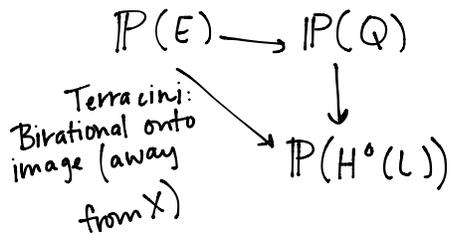
$$\begin{array}{ccc} X^{[n]} & \xrightarrow{\varphi} & G \\ & \searrow N_{n,L} & \downarrow d\varphi + \mathcal{Q} \\ & & \mathbb{P}(\Lambda^n H^0(L)) \end{array}$$

$\wedge^n H^0(L) \rightarrow H^0(N_{n,L}) \Rightarrow$ map is determined by complete linear system. Since it's an isomorphism, it's non-degenerate.

Note: φ is injective \Leftrightarrow no two length n subschemes determine the same n -plane $\Leftrightarrow L$ n -very ample

In fact, in this case φ is an embedding

Roundabout argument:



So $\mathbb{P}(E) \rightarrow \mathbb{P}(Q)$ is an embedding $\Rightarrow \varphi$ is an embedding